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ABSTRACT

The author describes a way of teaching a one-year geometry course which includes plane geometry, solid geometry, coordinate methods of proof, and vector methods of proof. His method is based upon encouraging students to use the properties of geometric figures that they already accept without formal proof. He employs paper folding extensively to arrive at many of the plane geometric concepts. (CT)

GEOMETRY IN SECONDARY SCHOOLS

by Bruce E. Meserve, University of Vermont

Based upon a presentation at the Atlanta, Georgia meeting of the National Council of Teachers of Mathematics, November 21, 1970.

Geometry in secondary schools has evolved during the last few years so that the use of coordinates has become respectable and the use of vectors is beginning to gain acceptance. There still remains the basic problem of finding sufficient time to consider the topics from the traditional full year of plane geometry, an introduction to the geometry of three space, an introduction to coordinate methods of proof, and an introduction to vector methods of proof.

My aim is to propose a procedure for solving this critical time shortage. The key to this proposed solution is the extensive previous knowledge that students have gained through their contacts with informal geometry both inside school and outside school.

Why should we belabor properties of geometric figures that the students already accept? In general, we hesitate to encourage students to use their previous knowledge because we want to present geometry as a mathematical system, a body of knowledge that is completely contained in a "mathematical game" with its own set of rules, or assumptions. Accordingly, it is essential that students know exactly what facts they are entitled to assume, that is, use without proof. Then the students are able to identify unequivocally the statements to be proved.

The procedure that I am suggesting is basically a suggestion to you as teachers, a pedagogical approach. The aim is to find an interesting way of recognizing explicitly the vocabulary and statements that the students should assume from their previous experiences in informal geometry.

Most people who have a piece of paper and don't have a pencil soon start doodling by making folds (creases) in the paper. We can think of each crease as a line. We assume that:

> There exists a straight line.

Intersections of creases (lines) can represent points. We develop this concept informally and assume that:

> If m is a line, then m is a set of points and may be used as a real number line.

Informal discussions suffice to enable students to recall, and recognize as acceptable the vocabulary of a geometric figure (set of points), half-line, ray, endpoint of a ray, line segment, endpoints of a line segment, the determination of a line segment by its endpoints, the determination of a line by any two of its points, betweenness for points on a line, interior point of a line segment, collinear points, noncollinear points, the determination of a plane by any three of its noncollinear points, and coplanar figures.

Several stages are needed in many of these discussions and the general content can arise in a variety of ways. However, the point of the procedure is that the students have the experience to develop the concepts that are to be recognized as acceptable for future use and therefore to be assumed without further proof. Here are a few more examples of conclusions that can be reached in discussions with students.

On a sheet of paper consider two opposite rays with a given point as their common endpoint. Fold one ray onto the other. The crease represents the line that is perpendicular to the line of the two rays at their common end point. On a given plane there is one and only one line perpendicular to a given line at a given point.

On a sheet of paper consider a line and a point that is not on the line. Fold the line onto itself so that the given point is on the fold. The crease represents the line that is perpendicular to the given line and contains the given point. On a given plane there is one and only one line perpendicular to a given line and containing a given point.

Informal discussions of the figures on two intersecting lines lead to angles, vertical angles, right angles, and adjacent angles. The congruence of vertical angles can be illustrated by folding one angle onto the other. The crease obtained in this way contains the bisectors of the other pair of vertical angles. If one of this second pair of vertical angles is folded onto the other, the second crease is perpendicular to the first. A variety of interesting theorems can be identified and a test developed for determining whether or not the lines represented by two intersecting creases are perpendicular.

At this stage it seems worth while to leave paper folding momentarily to discuss, or review, measures of angles, the angle-measure postulate, the protractor postulate, the addition property of angles, complementary angles, and supplementary angles. The usual theorems can be proved very easily since the students have the concepts from paper folding. The identification of acute angles, obtuse angles, and right angles can be done by having the students describe given angles and develop their own definitions. Interior points and exterior points of angles should be discussed. Whether a straight angle is admitted as an angle or called a pair of opposite rays is up to you. Personally, I consider it much better to recognize that a straight angle has no interior points rather than to deny its existence until the next year when it is needed for trigonometry.

An introduction of measures of line segments, congruence of line segments, the addition property of line segments, and midpoints is also desirable before proceeding with the recall of more properties of geometric figures from the students' experience in informal geometry.

Let us now consider two coplanar lines represented by creases in a piece of paper. The creases may intersect on the piece of paper. If the creases do not intersect on the piece of paper, we may or may not feel confident that they would intersect if the paper were larger. Students should be encouraged to find a way to construct lines that would not appear to intersect even on a very large sheet of paper. The previous work on perpendicular lines usually suffices to lead students to construct two lines perpendicular to the same line. After that the observation soon arises that two coplanar lines fail to intersect (are parallel) if and only if a line that is perpendicular to one of them is perpendicular to the other also.

We next consider three coplanar lines. If the lines are not concurrent (on a common point) and no two of the lines are parallel, then the three lines have a triangle as a subset. A discussion of angles and their interiors leads to the identification of the vertices of the triangle, angles of the triangle, interior points of the triangle, the triangular region, and exterior points of the triangle. The identification of various types of triangles can be done by having the students describe given triangles and develop their own definitions. The various conditions for congruent triangles can be assumed from the ways in which students have previously copied triangles using straightedge and compasses. Especially if the student doesn't recall one or more of the usual constructions, the construction should be repeated and the assumed congruence illustrated by superposition.

We have seen that explorations with paper folding and constructions can provide motivation for assuming several of the usual postulates and also many statements that are usually proved as theorems. The recognition and acceptance of terminology and properties of geometric figures from the students' informal experiences in geometry make it possible to find time for some of the other topics that should be included and which will be much more meaningful to the students than a formal rehash of things that they already know. First and foremost among these new topics is a discussion of logical concepts.

Students now explain statements in elementary school and prove statements in algebra. Then in geometry the students are ready for explicit consideration of compound statements using "and", "or", "if-then"; truth values of statements; truth tables; Venn diagrams; forms of statements of implication, negation (denial) of a statement; the converse, inverse, and contrapositive of a statement of implication; biconditional statements; rules of inference; applications of rules of inference to obtain proofs; direct proofs; indirect proofs; disproofs (proofs that a statement is false, usually by a counterexample); existential quantifiers; and universal quantifiers.

The emphasis upon informal approaches can now be shifted from providing a basis for the recognition of assumptions to providing a basis for the observations (conjectures) that are to be proved or disproved. For these proofs a wide variety of approaches will be sought — direct synthetic proofs, indirect proofs, disproofs by counterexample, coordinate proofs, vector proofs.

Consider the problem of identifying the locus of points that are equidistant from two given points. On a plane we may consider two points A and B on a piece of paper and fold one onto the other. The crease represents a line. Each point of this line is at the same distance from the two points A and B (now folded together). Informal discussions can lead to the identification of this line as containing the midpoint of $\overline{AB}$ and being perpendicular to $\overleftrightarrow{AB}$. Then the usual synthetic proofs can be used to prove that on the plane any point that is equidistant from A and B is on the perpendicular bisector of $\overline{AB}$. Similarly, any point of the perpendicular bisector of $\overline{AB}$ is equidistant from A and B. The bisector of an angle may be discussed in a similar manner. In this case fold one side of the angle onto the other. If the sides of the angle are not on the same line, the bisector of the angle is a ray with the vertex of the angle as endpoint. All other points of the angle bisector are interior points of the angle and are equidistant from the sides of the angle.

Good class discussions can be expected for the construction by paper folding of right triangles, isosceles triangles, right-isosceles triangles, and equilateral triangles, using two-thirds of a right angle. Practically all of the usual theorems regarding isosceles triangles can be conjectured from triangles represented by folds in a piece of paper. Then these conjectures can be proved.

Paper folding may also be used to construct lines associated with any triangle, not necessarily isosceles. For example, there are lines that contain the bisectors of the angles of the triangle, the altitudes of the triangle, the medians of the triangle, and the perpendicular bisectors of the sides of the triangle. Also lines may be constructed through each vertex of the triangle and parallel to the opposite side.

Represent any isosceles triangle by creases in a piece of paper and construct the three lines that contain the perpendicular bisectors of the sides of the triangle. Try this for several isosceles triangles and develop a criterion for determining in advance whether the constructed lines will appear to intersect at an interior point of the triangle, at a point of the triangle, or at an exterior point of the triangle. Notice that the existence of a triangle for which the lines appear to intersect at an exterior point disproves any conjecture that the lines always intersect at an interior point.

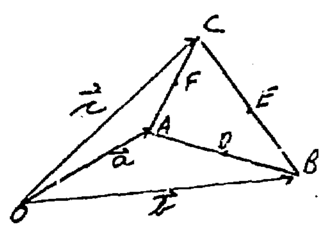
Try triangles of many shapes to try to find a counterexample for the concurrent appearance of the lines that contain the perpendicular bisectors of the sides. Students are now ready for the usual synthetic proof that the perpendicular bisectors of the sides of any triangle are concurrent. Since the students have already considered coordinate planes in algebra, at most the midpoint formula and slopes of lines would be needed to consider a coordinate proof. The vertices of the triangle may be chosen as A: (0, 0), B: (b, 0) where $0 < b$, and C: (c, d) where $0 < d$. The coordinate proof can be easily understood by the students.

If we construct the lines containing the bisectors of the angles of any triangle, these lines appear to be concurrent at an interior point of the triangle. Can you find a triangle such that the common point of the angle bisectors appears to be a point of the triangle or an exterior point? The usual synthetic proof that these lines are concurrent is very similar to the proof for the perpendicular bisectors of the sides of the triangle. Explorations can show that except for its endpoint each angle bisector is a subset of the interior of the angle and the interior of the triangle is the intersection of the interiors of its angles. Thus the common point of the angle bisectors must be an interior point of the triangle.

If we construct the lines containing the altitudes of a triangle, these lines also appear to be concurrent. The usual proof of this conjecture involves the construction of a second triangle with its sides on lines through each vertex and parallel to the opposite side of the given triangle. A coordinate proof for any triangle ABC with vertices at A: (0, 0), B: (b, 0), and C: (c, d) seems much easier than the synthetic proof.

If we construct the lines containing the medians of a triangle, these lines also appear to be concurrent. The usual synthetic proof involves the construction of a parallelogram. Although the proof can be followed by students, the structure is not immediately obvious to a beginner until after he has seen it. Even then the approach seems somewhat of an ingenious device. In contrast a coordinate proof is completely straightforward and the structure of the proof is clear to the student from the beginning. The coordinates of the vertices of the triangle are specified. The coordinates of the midpoints of the sides are found. The equations of the lines that contain the medians are found. Two of these equations are solved simultaneously and the solution is substituted in the third equation.

In the coordinate proof that the medians of any triangle are concurrent there are no devious devices or constructions. The coordinate proof is a straightforward application of established principles and the student knows where he is headed at all times. Such an approach leads to a new concept of secondary school geometry. The proofs are business-like procedures for establishing or disproving conjectures. Ingenuity is needed in making conjectures rather than in finding useful auxiliary lines or constructions.



Vector proofs are also straightforward. For example, here is a vector proof that the medians of any triangle ABC are concurrent. Select any point O. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. Then $\overrightarrow{AB} = -\vec{a} + \vec{b}$. If D is the midpoint of $\overline{AB}$, then

$$\overrightarrow{OD} = \vec{a} + \frac{1}{2}\overrightarrow{AB} = \vec{a} + \frac{1}{2}(-\vec{a} + \vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}.$$

Similarly for the midpoints E of $\overline{BC}$ and F of $\overline{AC}$ we have

$$\overrightarrow{OE} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}, \qquad \overrightarrow{OF} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{c}.$$

On the line $\overleftrightarrow{AE}$ we may think of A as the origin, E as the unit point, and each point R of the lines as having coordinate r. Then $\overrightarrow{AR} = r\,\overrightarrow{AE}$ and each point R of the line $\overleftrightarrow{AE}$ may be represented in this way relative to A. With reference to the point O, we have

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OA} + \overrightarrow{AR}\\ &= \vec{a} + r\left[-\vec{a} + \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}\right]\\ &= (1 - r)\,\vec{a} + \frac{r}{2}\vec{b} + \frac{r}{2}\vec{c}.\end{aligned}$$

Similarly, for each point S on $\overleftrightarrow{BF}$

$$\overrightarrow{OS} = \overrightarrow{OB} + \overrightarrow{BS}$$
$$= \vec{b} + s\left[-\vec{b} + \frac{1}{2}\vec{a} + \frac{1}{2}\vec{c}\right]$$
$$= \frac{s}{2}\vec{a} + (1 - s)\vec{b} + \frac{s}{2}\vec{c}.$$

At the point P of intersection of $\overrightarrow{AE}$ and $\overrightarrow{BF}$ the values of r and s must be such that

$$(1 - r)\vec{a} + \frac{r}{2}\vec{b} + \frac{r}{2}\vec{c} = \frac{s}{2}\vec{a} + (1 - s)\vec{b} + \frac{s}{2}\vec{c}.$$

Then since this relation is to hold for all triangles and thus for many vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, the coefficients of each vector must be equal; that is,

$$1 - r = \frac{s}{2}, \quad \frac{r}{2} = 1 - s, \quad \frac{r}{2} = \frac{s}{2}.$$

Then $r = s = \frac{2}{3}$ and $\overrightarrow{OP} = \frac{2}{3}(\vec{a} + \vec{b} + \vec{c})$.

For each point T on $\overleftrightarrow{CD}$ we have $\overrightarrow{OT} = \frac{t}{2}\vec{a} + \frac{t}{2}\vec{b} + (1 - t)\vec{c}$ and $P \in \overleftrightarrow{CD}$ for $t = \frac{2}{3}$. Since $r = s = t = \frac{2}{3}$ where $0 < \frac{2}{3} < 1$, the point P is on each of the line segments of the medians and is an interior point of the triangle.

Many theorems can be easily proved by coordinate and vector methods. For another example of a coordinate proof consider the theorem: Any angle inscribed in a semicircle is a right angle. The semicircle may be taken as the graph of $x^2 + y^2 = r^2$ where $0 \leq y$. Any angle inscribed in this semicircle is an $\angle ABC$ with A: $(-r, 0)$, B: (x, y), and C: $(r, 0)$ where $x^2 + y^2 = r^2$. Then $\overleftrightarrow{AB}$ has slope $y / (x + r)$; $\overleftrightarrow{BC}$ has slope $y / (x - r)$. The product of the slopes is -1 and the angle is a right angle. This property of angles enables us to use paper folding to construct as many points as we wish of a circle with a given diameter.

In this decade of the 70's we should expect secondary school geometry to make increasing use of student explorations and conjectures, to stress logical concepts without becoming more formal, to welcome coordinate proofs and vector proofs as well as others, and to treat geometry as a part of mathematics. In the study of mathematics algebraic concepts such as coordinates and vectors should be used freely in discussions of geometry. For example, the prismoidal formula may be used to obtain volumes of a wide variety of solids.[1] Similarly, geometric concepts should be used freely in the study of algebra, elementary functions, and other mathematical topics. For example, the intersectional properties of lines on a plane and planes in space are used in the study of systems of linear equations in algebra, conic sections are studied in algebra, and sample spaces are used in the study of probability. One of the most striking applications of geometry to the study of algebra is the use of an affine plane to show a relationship between the classification of roots of a quadratic equation in one variable and the classification of conics.[2]

Through our concept of geometry as an approach to mathematics we should recognize that there is some geometric content in each year of school mathematics. Our concept of geometry in secondary schools should include this broad concept of geometry as a basic approach to mathematics, a point of view for visualizing and understanding aspects of practically all mathematical subjects, and a significant part of each and every year of our school mathematics program.

1. Meserve, B.E. and R.E. Pinary, "Some Notes on the Prismoidal Formula," Mathematics Teacher, April, 1952, pp. 257-263.

2. Meserve, B.E. and M.A. Sobel, Mathematics for Secondary School Teachers, Prentice-Hall, Inc. 1962, pp. 303-306.